Spin-Hall conductivity in a two-dimensional Rashba electron gas

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Spin-Hall conductivity and Pauli susceptibility of 2D electron gas with Rashba spin-orbital interaction is studied theoretically in the semiclassical limit $k_F l \gg 1$. Static spin-Hall conductivity is shown to be zero for any nonvanishing disorder strength in the general case of the momentum-dependent Rashba velocity $\alpha(p)$ and nonparabolic spectrum $\epsilon(p)$. This result is derived both by an explicit diagrammatic calculation for the model of noninteracting electrons in a disorder potential, and via the analysis of general operator commutation relations, that are valid also for the case of interacting electrons. For the clean limit $l \to \infty$ and in the presence of electron-electron interactions, we derived the universal relation between frequency-dependent spin-Hall conductivity $\sigma_{\mathrm{sH}}(\Omega)$ and Pauli susceptibility $\chi(\Omega)$. Electron-electron interaction is shown to modify the "universal" value $\sigma_{\mathrm{sH}}^{(0)} = e/8\pi\hbar$ by the corrections of the relative magnitude determined by the standard Coulomb parameter only.

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I. INTRODUCTION

A *dissipationless* spin current can be generated in response to an electric field in semiconductors with the spin-orbital interaction.[1] For the case of an ideal two-dimensional (2D) electron gas with the Rashba coupling, Sinova *et al.*[2] have found a spin-Hall current of the transverse (z) spin component as a response to an in-plane electric field E_ν, $j_\mu^z = \sigma_{\mathrm{sH}} \epsilon_{\mu\nu} E_\nu$, with the "universal" spin-Hall conductivity

$$\sigma_{\mathrm{sH}} = \frac{e}{8\pi\hbar} \tag{1}$$

independent of the Rashba interaction constant α and density n, provided that both spin-split bands are occupied. This is the case when the density $n > n^* = m^2\alpha^2/\pi$.

The result (1) is drastically modified by the presence of disorder: for the standard Rashba model it was demonstrated that dc spin-Hall conductivity vanishes even in the case of an arbitrary weak disorder.[3–6] Although now there is no doubt about validity of this result, its physical origin does not seem to be completely clear, as it comes in calculations in the form of a somewhat mysterious cancellation between two nonvanishing contributions. In particular, it is not *a priori* evident if the same cancellation takes place for the generalized Rashba model with a momentum-dependent spin-orbital coupling constant $\alpha(p)$ and a general non-parabolic spectrum $\epsilon(p)$. Note that both these generalizations would lead to the lack of the special symmetry of the standard Rashba model, leading to equal values $v_+ = v_-$ of the Fermi velocities of the two chiral branches. In this paper we provide a fully microscopic diagrammatic calculation of the spin-Hall conductivity for the generalized model of a non-parabolic spectrum and an arbitrary momentum dependence of the Rashba velocity (3). We show that in the static limit $\sigma_{\mathrm{sH}} = 0$ *independently* upon the relation between the inverse elastic scattering time $1/\tau$ and the spin-orbital band splitting $\Delta = \alpha p_F$, similarly to the standard Rashba model studied in Refs. 3–6. Generality of this result indicates that it might be possible to derive it using some very general arguments, and it is indeed the case: we show, using some operator commutation relations (valid for interacting electrons as well), that the absence of the static spin-Hall conductivity follows from the stationarity of the state of an electron system with an applied electric field, thus $\sigma_{\mathrm{sH}} = 0$ for any nonvanishing disorder.

In a very clean 2D electron gas the mean free path l may exceed the system size L, and it is worth analyzing the frequency-dependent spin-Hall response $\sigma_{\mathrm{sH}}(\Omega)$. Recently, Rashba demonstrated[7] a direct relation between $\sigma_{\mathrm{sH}}(\Omega)$ and the dielectric response function $\epsilon(\Omega)$ of a clean noninteracting 2DEG with a spin-orbital interaction. We derive below a universal relation between the frequency-dependent spin-Hall conductivity $\sigma_{\mathrm{sH}}(\Omega)$ of a clean 2DEG and its in-plane magnetic susceptibility $\chi_\parallel(\Omega)$, providing additional arguments in favor of the equilibrium nature of the spin-Hall constant

$$\sigma_{\mathrm{sH}}(\Omega) = \frac{2e}{(g\mu_B)^2 m_b} \chi_\parallel(\Omega), \tag{2}$$

where m_b is the band mass, μ_B is the Bohr magneton, and g is the Lande factor. The relation (2) is valid for any spin-independent electron-electron interactions, at any frequency and for any electron density n consistent with the use of a parabolic band spectrum, $\epsilon(p) = p^2/2m_b$. This relation (2) holds even in the case of a very low $n < n^*$, when only one chiral subband is populated and the result of Sinova *et al.*,[2] Eq. (1), is not applicable.

Finally, we calculate corrections to the spin-Hall conductivity that are due to two-particle electron-electron interaction, and find these corrections to be nonzero. A direct microscopic calculation to the first order in the interaction strength shows that the electron-electron interaction renormalizes both the spin-Hall conductivity and the in-plane spin susceptibility, while keeping relation (2) intact. Relative magnitudes of these corrections are proportional to the dimensionless Coulomb strength $e^2/\epsilon\hbar v_F$ and do not contain the spin-orbital subband splitting Δ.

The rest of the paper is organized as follows. Section II contains a microscopic calculation proving that $\sigma_{sH}=0$ in the presence of disorder, and a general proof of this result via operator commutation relations. Then in Sec. III we derive the universal relation (2) and calculate the interaction-induced corrections to $\sigma_{\rm sH}(\Omega)$. Our conclusions are presented in Sec. IV.

II. VANISHING OF dc SPIN-HALL CONDUCTIVITY IN THE PRESENCE OF DISORDER

A. Microscopic diagrammatic calculation

2D isotropic Rashba gas is an electron system with broken inversion symmetry. In this case an electric field perpendicular to the plane could arise. It has no effect on the electron orbital motion but it couples to the electron spin via a relativistic spin-orbit interaction known as the Rashba term.[8] The Hamiltonian of an electron consists of the kinetic energy term and the Rashba term

$$\hat{h}_{\alpha\beta}(\vec{p}) = \epsilon(p)\,\delta_{\alpha\beta} + \alpha(p)(\sigma^x_{\alpha\beta}\hat{p}_y - \sigma^y_{\alpha\beta}\hat{p}_x), \tag{3}$$

where $\hat{p}_\mu=-i\hbar\partial_\mu$ is the operator of the momentum of the electron, $\epsilon(p)$ is the band spectrum, $\alpha(p)$ is the Rashba velocity, σ^i $(i=x,y,z)$ are the Pauli matrices, and α, β are the spin indices. The Hamiltonian (3) can be diagonalized by the unitary matrix

$$U(\vec{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ ie^{i\varphi_{\mathbf{p}}} & -ie^{i\varphi_{\mathbf{p}}}\end{pmatrix}, \tag{4}$$

where $\varphi_{\mathbf{p}}$ is the angle between the momentum $\vec{p}$ of the electron and the x axis, with the eigenvalues

$$\epsilon_\lambda(\vec{p}) = \epsilon(p) - \lambda p\alpha(p). \tag{5}$$

The eigenvalues $\lambda=\pm1$ of the chirality operator and the momentum of the electron $\vec{p}$ constitute the quantum numbers of an electron state $(\vec{p},\lambda)$. Fermi circles of the Rashba gas with the different chiralities are split: $p_{F\pm}=p_F[1\pm\alpha(p_F)/v(p_F)]$, where Fermi momentum p_F solves the equation $\epsilon(p_F)=\mu$, where μ is the chemical potential; $v(p)=d\epsilon(p)/dp$ is the band velocity of the electron. The electron velocity in the chiral state is $\partial\epsilon_\lambda(\vec{p})/\partial p$. We assume $\alpha(p_F)\ll v(p_F)$, and neglect corrections of the order α/v. The spin-orbital splitting is then $\Delta=2p_F\alpha(p_F)$. The density of states on the two Fermi circles differs as $\nu_\pm=\nu_F(1\pm\alpha_F/v_F)$, where $\nu_F=p_F/2\pi v(p_F)$. Contrary to the case of the parabolic spectrum and the Rashba velocity independent on the momentum, for the generalized model (3) the Fermi velocities are different on the two Fermi circles $v_{F+}-v_{F-}=2\alpha_F(p_F/mv_F-1)-2p_F(d\alpha/dp)|_F$. In the following we use the units where $\hbar=1$.

In this section we consider the 2D ideal (noninteracting electrons) Rashba electron gas with the Hamiltonian

$$\hat{H}_R = \int \psi^\dagger_\alpha(\vec{r})\hat{h}_{\alpha\beta}(\vec{p})\psi_\beta(\vec{r})d^2\vec{r}, \tag{6}$$

at zero temperature. $\psi^\dagger_\alpha(\vec{r})$ and $\psi_\beta(\vec{r})$ are the electron creation and annihilation operators. Electromagnetic vector potential $\vec{A}$ couples to the orbital motion of the electron according to the transformation $\vec{p}\to\vec{p}-e\vec{A}/c$ in the Hamiltonian (3). Variation of the Hamiltonian (27) with respect to $\vec{A}$ gives the electric current operator $\hat{J}_\nu=\int\psi^\dagger_\alpha(\vec{r})(\hat{j}_\nu)_{\alpha\beta}\psi_\beta(\vec{r})d^2\vec{r}$, where the one-particle current operator reads (it is actually a velocity $\hat{j}_\nu=e\hat{v}_\nu$):

$$(\hat{j}_\nu)_{\alpha\beta}(\vec{p}) = e\left(v(p)\frac{p_\nu}{p}\delta_{\alpha\beta} + \frac{d[p_\mu\alpha(p)]}{dp_\nu}\epsilon^{zi\mu}\sigma^i_{\alpha\beta}\right), \tag{7}$$

with $\nu=x,y$ being the spatial index and $\epsilon^{zi\mu}$ being the 3D totally antisymmetric tensor.

Under the nonuniform SU(2) electron spinor transformation $\psi_\alpha(\vec{r})\mapsto U_{\alpha\beta}(\vec{r})\psi_\beta(\vec{r})$, the Hamiltonian (27) becomes dependent on the SU(2) "spin electromagnetic" vector potential $\hat{A}_\mu=A^0_\mu\sigma^0+A^i_\mu\sigma^i$, where A^0_μ coincides with the physical electromagnetic potential and $A^i_\mu=-i\,\mathrm{Tr}(\sigma^i U^+\partial_\mu U)/2$. Although this latter potential is a pure gauge and has no physical consequences, variation of the Hamiltonian (27) with respect to it defines the spin current of the i component of the spin along the direction μ: $\hat{J}^i_\mu=\int\psi^\dagger_\alpha(\vec{r})(\hat{j}^i_\mu)_{\alpha\beta}\psi_\beta(\vec{r})d^2\vec{r}$, where the one-particle spin current operator reads

$$(\hat{j}^i_\mu)_{\alpha\beta}(\vec{p}) = v(p)\frac{p_\mu}{p}\sigma^i_{\alpha\beta} + \frac{d[p_\nu\alpha(p)]}{dp_\mu}\epsilon^{zi\nu}\delta_{\alpha\beta}. \tag{8}$$

Our definition of the spin current (8) coincides with the definition followed in Ref. 9: $\hat{J}^i_\mu=(\hat{v}_\mu\sigma^i+\sigma^i\hat{v}_\mu)/2$, but differs from the definition followed in Refs. 1–4 by a factor 2, which makes our value of the spin-Hall conductivity twice as large as that in the literature.[1–4]

The interaction of an electron with short-ranged nonmagnetic impurities at positions $\vec{R}_i$, numerated by the index i, is described by the impurity Hamiltonian

$$\hat{H}_{\rm imp} = \sum_i \int u(\vec{r}-\vec{R}_i)\psi^\dagger_\alpha(\vec{r})\psi_\alpha(\vec{r})d^2\vec{r}, \tag{9}$$

where $u(\vec{r})$ is a short-range impurity potential. We assume it to be sufficiently weak in order for the Born approximation to be valid. In this limit the impurity model (9) is equivalent to the model of the Gaussian random potential. We expand the electron Green's function averaged over the realizations of the disorder potential perturbatively in power of the Hamiltonian (9) using the diagrammatic procedure.[10] It is a sum of diagrams where a chain of electron bare Green's functions is separated by impurity "crosses." Two "crosses" are connected by the averaged impurity line $n_{\rm imp}u^2=1/2\pi\nu\tau$, where $n_{\rm imp}$ is the density of impurities, and τ is the scattering mean free time. A "cross" does not change the electron spin and the electron frequency since the electron scattering off impurity is elastic. Therefore the impurity line carries zero frequency. Diagrams with crossings of two or more impurity lines are small as powers of the ratio $1/\epsilon_F\tau\ll1$. The averaged Green's function is a two by two matrix in the spin space and it is a solution of the Dyson equation

$$G^{-1}_{\alpha\beta}(\epsilon,\vec{p}) - (\epsilon-\mu)\,\delta_{\alpha\beta} + h_{\alpha\beta}(\vec{p}) = -\frac{n_{\mathrm{imp}}u^2}{V}\sum_{\vec{p}'} G_{\alpha\beta}(\epsilon,\vec{p}'). \tag{10}$$

It can be conveniently transformed into the chiral basis by the unitary matrix $U(\vec{p})$ (4). The retarded and the advanced averaged Green's functions are diagonal in the chiral basis $G^{(R,A)}_{\lambda'\lambda}(\epsilon,\vec{p})=G^{(R,A)}_{\lambda}(\epsilon,\vec{p})\delta_{\lambda'\lambda}$, and the solution to the Dyson Eq. (10) reads[10]

$$G^R_\lambda(\epsilon,\vec{p}) = \frac{1}{\epsilon-\epsilon_\lambda(\vec{p})+\mu+i/2\tau}\delta_{\lambda'\lambda}, \tag{11}$$

where τ is explicitly independent of the chirality. The advanced Green's function is a complex conjugate of the retarded one: $G^A_\lambda(\epsilon,\vec{p})=\{G^R_\lambda(\epsilon,\vec{p})\}^*$. The Green's function in the spin basis is a nondiagonal two by two matrix:

$$G^{(R,A)}(\epsilon,\vec{p}) = \begin{pmatrix} G^{(R,A)}_{\uparrow\uparrow}(\epsilon,\vec{p}) & G^{(R,A)}_{\uparrow\downarrow}(\epsilon,\vec{p}) \\ G^{(R,A)}_{\downarrow\uparrow}(\epsilon,\vec{p}) & G^{(R,A)}_{\downarrow\downarrow}(\epsilon,\vec{p}) \end{pmatrix}, \tag{12}$$

where (omitting for a moment the frequency and momentum notations)

$$G^{(R,A)}_{\uparrow\uparrow} = G^{(R,A)}_{\downarrow\downarrow} = (G^{(R,A)}_{+} + G^{(R,A)}_{-})/2,$$

$$G^{(R,A)}_{\uparrow\downarrow} = -\,ie^{-i\varphi_{\mathbf{p}}}(G^{(R,A)}_{+} - G^{(R,A)}_{-})/2,$$

$$G^{(R,A)}_{\downarrow\uparrow} = ie^{i\varphi_{\mathbf{p}}}(G^{(R,A)}_{+} - G^{(R,A)}_{-})/2, \tag{13}$$

with the chiral $G^{(R,A)}_{\pm}$ being defined in Eq. (11).

In order to calculate the current, induced in the electron system by an electric field, we use the Keldysh technique.[11] Our result given by Eq. (16) is well known but we derive it here for consistency. The averaged Keldysh Green's function is a four by four matrix $\mathcal{G}(\epsilon,\vec{p})$ that can be conveniently factorized into a two by two Keldysh matrix whose elements are matrices in the spin space themselves:

$$\begin{pmatrix} \mathcal{G}_{--} & \mathcal{G}_{-+} \\ \mathcal{G}_{+-} & \mathcal{G}_{++} \end{pmatrix} = \begin{pmatrix} 1-N(\epsilon) & -N(\epsilon) \\ 1-N(\epsilon) & -N(\epsilon) \end{pmatrix} G^R(\epsilon,\vec{p}) + \begin{pmatrix} N(\epsilon) & N(\epsilon) \\ -1+N(\epsilon) & -1+N(\epsilon) \end{pmatrix} G^A(\epsilon,\vec{p}), \tag{14}$$

where the electron distribution $N(\epsilon)$ is proportional to the unit matrix in the spin space.

We choose the gauge for the uniform electric field $\vec{E}(t) = \vec{E}(\Omega)e^{-i\Omega t}$ to be a time-dependent vector potential $\vec{A}(t) = \vec{A}(\Omega)e^{-i\Omega t}$, where $\vec{A}(\Omega)=-ic\vec{E}(\Omega)/\Omega$. Using the Keldysh technique,[11] we average the spin current operator over the electron state perturbed by the electromagnetic Hamiltonian $\hat{H}_{\mathrm{em}}=-(1/c)\int d^2\vec{r}\hat{j}_\nu(\vec{r})A_\nu(t)$, in the first order of the perturbation theory. The spin-Hall conductivity σ_{sH} is then found from the relationship $\langle\hat{j}^z_\mu(\Omega)\rangle=\epsilon_{\mu\nu}\sigma_{\mathrm{sH}}(\Omega)E_\nu(\Omega)$ as

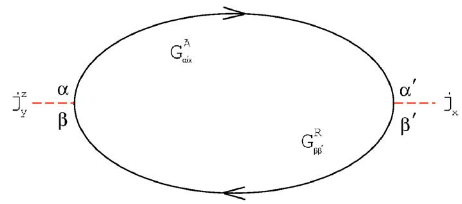

FIG. 1. The spin-Hall conductivity given by a one-loop diagram.

$$\sigma_{\mathrm{sH}} = \frac{-1}{V\Omega}\sum_{\vec{p}}\int\frac{d\epsilon}{2\pi}\mathrm{Tr}[\hat{j}^z_y(\vec{p})\mathcal{G}(\epsilon+\Omega,\vec{p})\tau^z\hat{j}_x(\vec{p})\mathcal{G}(\epsilon,\vec{p})]_{-+}, \tag{15}$$

where τ^z is the four by four matrix given by the direct product of the Pauli matrix σ^z in the Keldysh space and the unit matrix in the spin space, the current operators in Eq. (15) are the direct product of matrices (7) and (8) and the unitary matrix in the Keldysh space. Tr in Eq. (15) operates only in the spin space whereas the indices of $-+$ element corresponds to the Keldysh space. Substituting the Green's function Eq. (14), we obtain

$$\sigma_{\mathrm{sH}}(\Omega) = \frac{1}{V\Omega}\sum_{\vec{p}}\int\frac{d\epsilon}{2\pi}\langle\mathrm{Tr}\{\hat{j}^z_y(\vec{p})[N(\epsilon+\Omega) \times(G^R(\epsilon+\Omega,\vec{p})-G^A(\epsilon+\Omega,\vec{p}))\hat{j}_x(\vec{p})G^A(\epsilon,\vec{p}) + G^R(\epsilon+\Omega,\vec{p})\hat{j}_x(\vec{p})N(\epsilon)(G^R(\epsilon,\vec{p})-G^A(\epsilon,\vec{p}))]\}\rangle, \tag{16}$$

where the brackets indicate averaging over the disorder. In the noncrossing approximation the average in Eq. (16) is given by the sum of the one-loop and the ladder diagrams shown in Figs. 1 and 2.

First, we calculate the one-loop diagram in Fig. 1 and denote this part of the spin-Hall conductivity as σ^0_{sH}. It corresponds to Eq. (16) with all Green's functions being substituted by the averaged Green's functions (11). The second line in Eq. (16) contains the imaginary part of the Green's function $G^R(\epsilon+\Omega,\vec{p})$ and the corresponding integral is convergent, therefore we change ϵ to $\epsilon-\Omega$ in this second line. At $T=0$ the Fermi-Dirac distribution function reads: $N(\epsilon) = \theta(-\epsilon)$. We take the integral over ϵ and in the zero-frequency limit $\Omega\to 0$ we find

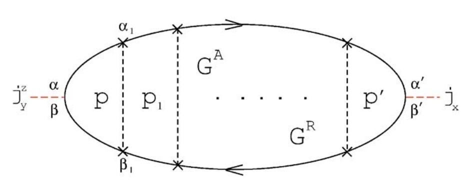

FIG. 2. The vertex correction to the spin-Hall conductivity is given by a sum of noncrossing ladder diagrams.

$$\sigma_{\mathrm{sH}}^{0} = -\frac{1}{V}\sum_{\vec{p}} \frac{v(p)}{2\pi p}\left[\frac{1}{2p\alpha(p)}[S(\zeta) - S(\xi)] + \frac{1}{2\tau}\frac{\zeta\xi + 1/4\tau^2}{(\zeta^2 + 1/4\tau^2)(\xi^2 + 1/4\tau^2)}\right], \tag{17}$$

where $S(x)=\arctan(x/2\tau)$, $\zeta=\epsilon(p)-p\alpha(p)-\mu$ and $\xi=\epsilon(p)+p\alpha(p)-\mu$. In the large volume limit we substitute $(1/V)\Sigma_{\vec{p}}\to\int[d^2\vec{p}/(2\pi)^2]$, and then evaluate the integral over $\vec{p}$ in Eq. (17) in the limit of large Fermi circle $\mu\gg\Delta$, $1/\tau$. This procedure, known as the semiclassical approximation, expresses the momentum p in terms of the quasiparticle energy ξ:

$$\int \frac{d^2\vec{p}}{(2\pi)^2} \approx \frac{p}{2\pi v(p)}\int_{-\infty}^{\infty}[1 + R(p)\xi]d\xi\int_0^{2\pi}\frac{d\varphi}{2\pi}, \tag{18}$$

where $R(p)=p^{-1}v^{-1}(p)-m^{-1}(p)v^{-2}(p)$. The result reads

$$\sigma_{\mathrm{sH}}^{0} = \frac{e}{4\pi}\left(1 - \frac{1}{1 + (\Delta\tau)^2}\right), \tag{19}$$

with the first and the second terms corresponding to the first and the second terms of Eq. (17), respectively.

An important observation is that the result (19) coincides exactly with the result obtained from those terms in Eq. (16) that contains one retarded and one advanced Green's function. These two terms are proportional to $dN(\epsilon)/d\epsilon=-\delta(\epsilon)$ and all integrals are explicitly confined to the vicinity of the Fermi circle. Therefore the spin-Hall conductivity, unlike the usual Hall conductivity, is determined by the quasiparticles around the Fermi circle and not by the entire Fermi disk.

The ladder diagrams shown in Fig. 2 represent the vertex corrections to the current. Additional impurity lines improve the convergence of the integral in Eq. (16). As a consequence, vertex corrections to the terms in Eq. (16) with the two advanced or with the two retarded Green's functions vanish as $\max(1/\tau,\Delta)/\epsilon_F\ll 1$. Therefore we consider only the vertex corrections to the terms with one advanced and one retarded Green's function as it is shown in Fig. 2. For these diagrams the semiclassical approximation (18) is valid.

The sum of ladder diagrams with $n=1,2,\ldots,$ impurity lines is given by the expression:

$$\sigma_{\mathrm{sH}}^{\mathrm{lad}} = -\int \frac{d^2\vec{p}}{(2\pi)^2}\mathrm{Tr}[\tilde{J}_y^z G^R(0,\vec{p}) j_x(\vec{p}) G^A(0,\vec{p})], \tag{20}$$

where the sum of $n=1\cdots\infty$ vertex corrections to the current $\hat{j}_y^z(\vec{p})$ (with at least one impurity line) is denoted by the matrix $\tilde{J}_y^z$. In the spin basis and for short-ranged impurity potentials it does not depend on the electron momentum $\vec{p}$ and satisfies the transfer matrix equation

$$\tilde{J}_y^z = \frac{1}{2\pi\tau\nu}\int \frac{d^2\vec{p}}{(2\pi)^2} G^A(0,\vec{p})[j_y^z(\vec{p}) + \tilde{J}_y^z]G^R(0,\vec{p}), \tag{21}$$

where the Green's functions $G^{(R,A)}$ are given by Eq. (12). The "full" spin current operator with all vertex corrections included: $\hat{j}_y^z(\vec{p})+\tilde{J}_y^z$, is represented diagrammatically in Fig. 3. In the equations for the current operators (7) and (8) we

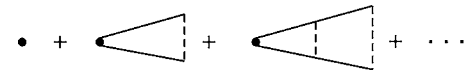

FIG. 3. The vertex of the spin current with the vertex corrections taken into account: $j_y^z(\vec{p})+\tilde{J}_y^z$.

expand the electron velocity $v(p)=v(p_F)+\xi/[v(p_F)m(p_F)]$, where $m^{-1}(p)=dv(p)/dp$, to the first order in the deviation from the Fermi circle: ξ/μ, small in the semiclassical approximation. We also expand the spin-orbital splitting $p\alpha(p)=\alpha(p_F)\{p_F+[1+(p_F/\alpha_F)d\alpha/dp_F]\xi/v_F\}$, in the Green's functions. We then evaluate Eq. (21) in the semiclassical approximation (18) neglecting odd powers of ξ:

$$(\tilde{J}_y^z)_{\uparrow\uparrow} = \{[2 + (\Delta\tau)^2](\tilde{J}_y^z)_{\uparrow\uparrow} + (\Delta\tau)^2(\tilde{J}_y^z)_{\downarrow\downarrow}\}B,$$

$$(\tilde{J}_y^z)_{\downarrow\downarrow} = \{(\Delta\tau)^2(\tilde{J}_y^z)_{\uparrow\uparrow} + [2 + (\Delta\tau)^2](\tilde{J}_y^z)_{\downarrow\downarrow}\}B,$$

$$(\tilde{J}_y^z)_{\uparrow\downarrow} = \{-iv(p_F)\Delta\tau + [2 + (\Delta\tau)^2](\tilde{J}_y^z)_{\uparrow\downarrow}\}B,$$

$$(\tilde{J}_y^z)_{\downarrow\uparrow} = \{iv(p_F)\Delta\tau + [2 + (\Delta\tau)^2](\tilde{J}_y^z)_{\downarrow\uparrow}\}B, \tag{22}$$

where $B=\frac{1}{2}[1+(\Delta\tau)^2]^{-1}$. From the first two lines of Eqs. (22) we find $(\tilde{J}_y^z)_{\uparrow\uparrow}=(\tilde{J}_y^z)_{\downarrow\downarrow}$, whereas from the last two lines we find $(\tilde{J}_y^z)_{\downarrow\uparrow}=iv(p_F)/(\Delta\tau)$ and $(\tilde{J}_y^z)_{\uparrow\downarrow}=-iv(p_F)/(\Delta\tau)$. The integrand of Eq. (20) does not depend on the diagonal elements of the matrix $\tilde{J}_y^z$ and therefore we set them to zero: $\tilde{J}_y^z=\sigma^y v(p_F)/\Delta\tau$. As it was expected the vertex corrections are proportional to the scattering rate. Integrating Eq. (20) in the semiclassical approximation (18) over ξ, we finally find the ladder part of the spin Hall conductivity (Fig. 2):

$$\sigma_{\mathrm{sH}}^{\mathrm{lad}} = \frac{e}{4\pi}\left(-1 + \frac{1}{1 + (\Delta\tau)^2}\right). \tag{23}$$

Remarkably all derivatives of $\alpha(p)$ and $v(p)$ over p have canceled out.

The spin-Hall conductivity is the sum of Eqs. (19) and (23) and is zero:

$$\sigma_{\mathrm{sH}} = \sigma_{\mathrm{sH}}^{0} + \sigma_{\mathrm{sH}}^{\mathrm{lad}} = 0. \tag{24}$$

It explicitly does not depend on the impurity scattering time τ. But, we observe a discontinuity between the spin-Hall conductivity in the clean system $\sigma_{\mathrm{sH}}=e/8\pi\hbar$ and the spin-Hall conductivity Eq. (24) in the presence of the infinitely small amount of nonmagnetic scatterers. As it was shown in Ref. 6, this discontinuity is related to the dissipation in the system, which gives rise to the dissipative part in the spin-Hall conductivity $\sigma_{\mathrm{sH}}^{D}=-e/8\pi\hbar$, which cancels the reactive part $\sigma_{\mathrm{sH}}^{R}=e/8\pi\hbar$.

In an analogous calculation, for the generalized model (3), we find the average spin polarization induced by the electric field

$$\langle \hat{S}^{\mu} \rangle = \epsilon^{\mu\nu} \frac{e\Delta\tau}{2\pi v_F} E_{\nu}, \tag{25}$$

in agreement with Refs. 4, 12, and 13. One should notice that the steady in-plane spin polarization (25) is a consequence of the zero spin-Hall effect (24). Nonzero spin Hall conductivity would result in a non-steady in-plane spin polarization.

B. General proof for the absence of the stationary spin-Hall current

Rashba, in his recent paper,[14] proves that the zero bulk spin-Hall conductivity is an intrinsic property of the free-electron Hamiltonian and scattering is merely a tool to reveal this property in terms of the diagrammatic technique. In this section we prove the zero spin-Hall conductivity via analysis of the general operator commutation relations. We start with the evolution equation of the total spin of the system $\hat{S}_{\nu}$ and the commutation relation

$$-i\frac{d}{dt}\hat{S}^{\mu} = [\hat{H},\hat{S}^{\mu}] = i\alpha(p)\frac{p}{v(p)}\hat{J}^{z}_{\mu}, \tag{26}$$

where $\mu=x,y$; $\hat{S}^{\mu}=\frac{1}{2}\int \psi^{\dagger}_{\alpha}(\vec{r})\hat{\sigma}^{\mu}\psi_{\beta}(\vec{r})d^2\vec{r}$ is the operator of the total spin of the electron system, $\hat{J}^{z}_{\mu}$ is the operator of the total spin current, defined in Eq. (8). Remarkably, Eq. (26) is valid for an electron system with any nonmagnetic disorder and any electron-electron interaction. If $\sigma_{\text{sH}} \neq 0$, then a homogeneous in space stationary spin current flows perpendicularly to the homogeneous in space electric field. Averaging the operator Eq. (26) over the stationary density matrix, we obtain a new equation for the c numbers instead of the operators: the time derivative of the total observable spin of the system is equal to the total average spin current. But the average total spin of a system is a limited quantity, thus such an equation cannot be valid for an infinitely long time as needed for a stationary response. This proves that either $\sigma_{\text{sH}}=0$ or the spin-Hall response in the Rashba metal is a nonstationary one. In the next section we investigate the second possibility which is realized naturally in the case of ac applied electric field.

III. SPIN-HALL CONDUCTIVITY AND PAULI SUSCEPTIBILITY IN THE PRESENCE OF ELECTRON-ELECTRON INTERACTIONS

A. Relation between the frequency-dependent spin-Hall conductivity and spin susceptibility

We consider the 2D interacting Rashba electron gas at zero temperature with the Hamiltonian

$$\hat{H} = \int \psi^{\dagger}_{\alpha}(\vec{r})\hat{h}_{\alpha\beta}(\vec{p})\psi_{\beta}(\vec{r})d^2\vec{r} + \frac{1}{2}\int\int \psi^{\dagger}_{\alpha}(\vec{r})\psi^{\dagger}_{\beta}(\vec{r}\,') \times U(|\vec{r}-\vec{r}\,'|)\psi_{\beta}(\vec{r}\,')\psi_{\alpha}(\vec{r})d^2\vec{r}d^2\vec{r}\,', \tag{27}$$

where $U(|\vec{r}|)$ is an arbitrary two-electron spin-independent interaction potential and $\hat{h}_{\alpha\beta}(\vec{p})$ is defined in Eq. (3). Essential for the following in this section is the Rashba velocity independent on momentum: $\alpha(p)=\alpha$, and the parabolic band spectrum

$$\epsilon(p) = \frac{p^2}{2m_b}. \tag{28}$$

Hamiltonian (27) is a rather accurate approximation for the clean two-dimensional semiconducting heterostructures.

To derive the relation (2) between the spin-Hall constant and the Pauli susceptibility, we start from two exact commutation relations for the total current operator and the total spin operator. For the assumed parabolic band spectrum (28), a certain linear combination of the total charge current $\vec{J}$ and the total spin $\vec{S}$ is proportional to the total momentum of the system and commutes with the interaction part of the Hamiltonian. This fact provides us with two exact commutation relations in the presence of an arbitrary spin-conserving two-particle interaction $U(|\vec{r}-\vec{r}\,'|)$ in the Hamiltonian (27):

$$[\hat{H},\hat{J}_{\nu}] = -2iem_b\alpha^2\epsilon^{\nu\mu}\hat{J}^{z}_{\mu}$$

and

$$[\hat{H},\hat{S}^{\mu}] = im_b\alpha\hat{J}^{z}_{\mu}, \tag{29}$$

where the total current and the spin current operators $\hat{J}_{\nu}$ and $\hat{J}^{z}_{\mu}$ are obtained from Eqs. (7) and (8), using Eq. (28).

The average spin current of the electron system as a response to weak ac electric field $E_x(t)=E_{0x}\cos\Omega t$, is given by the general quantum mechanical expression in the first order of perturbation theory[15]

$$\langle \hat{J}^{z}_{y}(t)\rangle = \frac{i}{2}\sum_{m}\left[(\hat{J}_x)_{m0}\left\{\frac{e^{-i\Omega t}}{\Omega(\omega_{m0}-\Omega-i0)} - \frac{e^{i\Omega t}}{\Omega(\omega_{m0}+\Omega-i0)}\right\} \times(\hat{J}^{z}_{y})_{0m} - \text{H.c.}\right]E_{0x}, \tag{30}$$

where $\omega_{m0}=\epsilon_m-\epsilon_0$, with 0 being the ground state, and m being the *exact* excitation levels of the interacting system. Note that we have used the Kubo formula (30) for external fields homogeneous in space.

Using the exact commutation relations (29), we can express the matrix elements of the total charge and spin current operators in the right-hand side of Eq. (30) in terms of the matrix elements of the total spin operator

$$\langle \hat{J}^{z}_{y}(t)\rangle = -\frac{e}{m_b}\sum_{m}\left[(\hat{S}^{y})_{m0}\left\{\frac{e^{-i\Omega t}}{\omega_{m0}-\Omega-i0} + \frac{e^{i\Omega t}}{\omega_{m0}+\Omega-i0}\right\} \times(\hat{S}^{y})_{0m} + \text{H.c.}\right]E_{0x}. \tag{31}$$

Note that now the right-hand side of Eq. (31) is fully analogous [up to a replacement of the e/m_b factor by $(g\mu_b)^2$] to the linear-response expression for Pauli spin susceptibility with respect to an in-plane magnetic field $H_y(t)=H_{0y}\cos\Omega t$, which would replace the electric field E_{0x}. This observation leads us immediately to the relation (2) which is the main result of the present paper. This relation holds, remarkably, for linear response to perturbations with an arbitrary frequency Ω which are uniform in space, i.e., $q=0$. In Ref. 16 a similar relation between the spin-Hall conductivity and the Pauli susceptibility is discussed for a noninteracting Rashba electron gas.

The Fermi liquid response function usually depends on the ratio Ω/qv_F. For example, in a normal isotropic Fermi liquid $\chi=0$ if the limit $q\to 0$, $\Omega\to 0$ is taken with the ratio $qv_F/\Omega\to 0$ as a consequence of the total spin conservation. The standard Pauli susceptibility $\chi_{\mathrm{Pauli}}=2\mu_B^2\nu(\epsilon_F)$ is obtained with the opposite order of limits, $\Omega/qv_F\to 0$ and $q\to 0$. In the case of the Rashba Fermi gas Gor'kov and Rashba[17] have found that $\chi_{zz}=\chi_{\parallel}=\chi_{\mathrm{Pauli}}$ at $\Omega/qv_F=0$ and $q\to 0$. We find $\chi_{\parallel}=\frac{1}{2}\chi_{\mathrm{Pauli}}$ and $\chi_{zz}=\chi_{\mathrm{Pauli}}$ at $qv_F/\Omega=0$ and $\Omega\to 0$. In relation (2) the zero-frequency limit of susceptibility $\chi(\Omega\to 0)=\frac{1}{2}\chi_{\mathrm{Pauli}}$ and spin-Hall response $\sigma_{\mathrm{sH}}(\Omega\to 0)=1/4\pi e$ is achieved via taking first of all the limit $q\to 0$ at nonzero Ω, and then decreasing Ω to zero. Therefore we expect the relation (2) to be valid for $qv_F/\Omega\ll 1$.

We checked by direct calculation for a clean system of noninteracting fermions of higher spin j, that spin susceptibilities and spin-Hall conductivities (32) follow relation (2). It is interesting that we find that in the case of an ideal 2D Rashba gas of fermions of arbitrary half-integer spin j the value of the spin-Hall constant is also universal and grows with j:

$$\sigma_{\mathrm{sH}}(j)=\frac{e}{2\pi}\sum_{m=-j}^{j}m^2. \tag{32}$$

B. Interaction corrections to the spin-Hall conductivity

In this subsection we calculate the interaction corrections to the spin-Hall conductivity (1) for a clean system in the zero-frequency limit: $\lim_{\Omega\to 0}\lim_{\tau\to\infty}\delta\sigma_{\mathrm{sH}}(\Omega)$. We use the Keldysh technique.[11] In the lowest order of the e-e interaction $U(|\vec{r}|)$, three diagrams, shown in Fig. 4, contribute to $\delta\sigma_{\mathrm{sH}}(\Omega)$.

We choose the gauge for the uniform electric field $\vec{E}(t)=\vec{E}(\Omega)e^{-i\Omega t}$ to be a time-dependent vector potential $\vec{A}(t)=\vec{A}(\Omega)e^{-i\Omega t}$, where $\vec{A}(\Omega)=-ic\vec{E}(\Omega)/\Omega$. Using the Keldysh technique we average the spin current operator over the electron state perturbed by both the electromagnetic Hamiltonian, $\hat{H}_{\mathrm{em}}=-(1/c)\int d^2\vec{r}\hat{j}_\nu(\vec{r})A_\nu(t)$, and the electron-electron interaction Hamiltonian $\frac{1}{2}\iint\psi_\alpha^\dagger(\vec{r})\psi_\beta^\dagger(\vec{r}')U(|\vec{r}-\vec{r}'|)\psi_\beta(\vec{r}')\psi_\alpha(\vec{r})d^2\vec{r}d^2\vec{r}'$, to the first order of the perturbation theory. The correction to the spin-Hall conductivity $\delta\sigma_{\mathrm{sH}}$ is then found from the relationship $\langle\hat{j}_\mu^z(\Omega)\rangle=\epsilon_{\mu\nu}[\sigma_{\mathrm{sH}}(\Omega)+\delta\sigma_{\mathrm{sH}}(\Omega)]E_\nu(\Omega)$. The resulting expression for the correction to the spin-Hall conductivity from the electron-electron interactions reads as follows:

$$\delta\sigma_{\mathrm{sH}}(\Omega)=\frac{e}{2V\Omega}\sum_{\vec{p},\vec{p}'}\int\frac{d\epsilon}{2\pi}\frac{d\epsilon'}{2\pi}\mathrm{Tr}[A+B+C]_{-+}U(|\vec{p}-\vec{p}'|),$$

where

$$A=J_y^z\mathcal{G}(\epsilon-\Omega,\vec{p})[\tau_z,\mathcal{G}(\epsilon'-\Omega,\vec{p}')\tau_zJ_x\mathcal{G}(\epsilon',\vec{p}')]_+\mathcal{G}(\epsilon,\vec{p}),$$

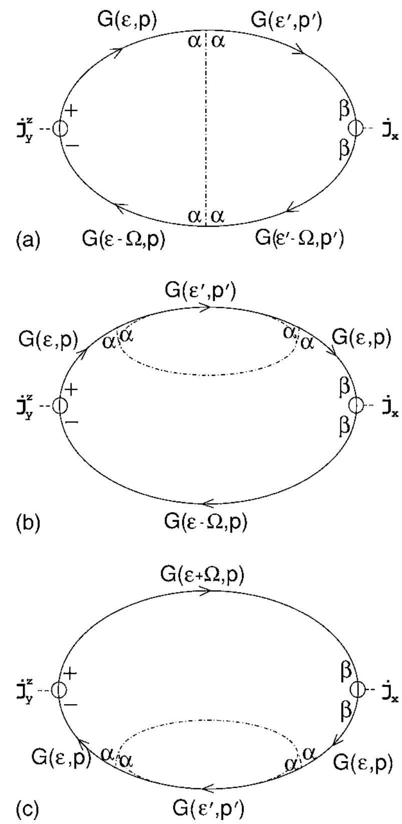

FIG. 4. The correction to the spin-Hall conductivity from the electron-electron interactions is given by the sum of the three diagrams, which have equal sign and coefficient. Indices $+$, $-$, α, β correspond to the Keldysh space. Dashed lines correspond to the interaction $U(|\vec{p}-\vec{p}'|)$.

$$B=J_y^z\mathcal{G}(\epsilon-\Omega,\vec{p})\tau_zJ_x\mathcal{G}(\epsilon,\vec{p})[\tau_z,\mathcal{G}(\epsilon',\vec{p}')]_+\mathcal{G}(\epsilon,\vec{p}),$$

$$C=J_y^z\mathcal{G}(\epsilon,\vec{p})[\tau_z,\mathcal{G}(\epsilon',\vec{p}')]_+\mathcal{G}(\epsilon,\vec{p})\tau_zJ_x\mathcal{G}(\epsilon+\Omega,\vec{p}). \tag{33}$$

τ^z is the four by four matrix given by the direct product of the Pauli matrix σ^z in the Keldysh space and the unit matrix in the spin space. The averaged Keldysh Green's function $\mathcal{G}(\epsilon,\vec{p})$ is a four by four matrix, defined in Eq. (14), where for this section we can choose the electron distribution $N(p)$ as a function of the momentum p due to the preserved translation invariance (no disorder), and for the advanced and the retarded Green's functions we set in Eq. (11) $\tau\to\infty$. The current operators in Eq. (33) are the direct products of the matrices (7) and (8) for the parabolic spectrum (28) and the unitary matrix in the Keldysh space. The Tr in Eq. (33) operates only in the spin space whereas the indices $-+$ correspond to the Keldysh space.

Calculation of the integrals in Eq. (33) in the limit $\Omega\to 0$ leads to the following result for the correction to the static conductivity:

$$\delta\sigma_{\mathrm{sH}}=-e\int\int\frac{d^2\vec{p}}{(2\pi)^2}\frac{d^2\vec{p}'}{(2\pi)^2}\delta N(p)\delta N(p')U(|\vec{p}-\vec{p}'|)F(\vec{p},\vec{p}').$$

Here

$$\delta N(p) = N_+(p) - N_-(p)$$

and

$$F(\vec{p},\vec{p}\,') = \frac{\cos(\varphi-\varphi')\{-p^2+pp'\cos(\varphi-\varphi')\}}{8m_b\alpha^2p^2p'^2}, \quad (34)$$

with $N_\pm(p)$ being the distribution functions of the two Fermi circles of different chiralities. For zero temperature $N_\pm(p) = \theta(-p+p_{F\pm})$. Our result (34) agrees with Ref. 18.

Explicit integration over momenta in the expression (34) was performed for small spin-orbit interaction $\alpha/v_F \ll 1$ in two limiting cases: short-ranged two-particle interaction (Coulomb potential screened on the length scale κ^{-1} smaller than interparticle distance), and full long-range Coulomb interaction. In the Fourier space these interaction potentials are: $U_1(|\vec{p}-\vec{p}\,'|)=2\pi e^2/\kappa\epsilon$, and $U_2(|\vec{p}-\vec{p}\,'|)=2\pi e^2/\epsilon|\vec{p}-\vec{p}\,'|$. The final expressions for $\sigma_{\rm sH}$ in these two cases are as follows:

$$\sigma_{\rm sH}^{\rm (short)} = \frac{e}{4\pi\hbar}\left[1-\frac{m_b e^2}{2\epsilon\kappa}\right] \quad (35)$$

for the short-range potential and

$$\sigma_{\rm sH}^{\rm (Coulomb)} = \frac{e}{4\pi\hbar}\left[1-\frac{2m_b e^2}{3\pi\epsilon p_F}\right] \quad (36)$$

for the Coulomb potential. It is seen that the correction to the spin-Hall conductivity is independent of the spin-orbit constant α [in Eq. (36) corrections of the order $(\alpha/v_F)^2 \ll 1$ are neglected], and is proportional to the standard Coulomb interaction parameter $e^2/\epsilon\hbar v_F$.

For completeness we have performed direct diagram calculations of the interaction correction to the in-plane susceptibility, represented by three diagrams similar to those shown in Fig. 1. The results for the relative corrections to the in-plane susceptibility were found to coincide with expressions (35) and (36), in agreement with the general relation (2).

IV. CONCLUSIONS

To conclude, we have generalized the result of Inoue *et al.* Ref. 3 $\sigma_{\rm sH}=0$ for the case of the arbitrary electron dispersion, arbitrary strength of disorder, and arbitrary momentum dependence of the Rashba velocity $\alpha(p)$, which extends considerably its applicability range. In particular, we found that vanishing of the spin-Hall conductivity in bulk samples with impurities is not limited to the specific case of equal Fermi velocities on different chiral branches. Our result Eq. (24) agrees with Refs. 4–6. Moreover, we provided general arguments for the zero static bulk spin-Hall conductivity in the presence of an arbitrary nonmagnetic disorder and arbitrary spin-conserving electron-electron interaction.

In the second part of the paper we have shown that the frequency-dependent spin-Hall conductivity and the Pauli susceptibility of a clean interacting 2D Rashba EG are proportional to each other, with the coefficient containing band mass, Lande factor and Bohr magneton only. We calculated the first-order interaction-induced correction to the spin-Hall conductivity and found it to be proportional to the standard dimensionless interaction strength. We expect our results for the clean Rashba electron gas to be directly relevant for submicron samples with size less than the elastic scattering length. The influence of disorder upon $\sigma_{\rm sH}$ in the presence of electron-electron interactions was studied in Ref. 18.

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